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WVT-7204

TRANSVERSELY ISOTROPIC BEAMS UNDER INITIAL STRESS

APRIL 1972

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BENÉT WEAPONS LABORATORY

WATERVLIET ARSENAL
Watervliet, New York

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BY

EUGENE J. BRUNELLE

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A linearized theory of transversely isotropic beams, which accounts for initial non-uniform states of stress, is derived by perturbing and averaging (thru the thickness) the non-linear equations of elasticity. This theory provides a rigorous foundation for previously derived linear equations as well as displaying some new features concerned with the stability and vibration of beams acted upon by a variety of initial stresses. A comparison is made to a previous theory (suitably one-dimensionalized) by Herrmann and Armenakas which was obtained by variational methods and, although the results are vaguely similar, surprising differences are found which support a previous criticism of Masur.

Cross-Reference Data

Nonlinear Elasticity

Initial Stresses

Beams

Vibration

Elastic Stability

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INTRODUCTION

During the past two decades significant advances have been made in the theory of non-linear elasticity. One of the potentially most useful areas of research in non-linear elasticity, from an engineering viewpoint, has been the development of small deformation equations superposed on a state of finite deformation [1]. A logical extension of these results would be to adapt these equations, by means of an averaging procedure, to describe beam and plate behavior. For example the spirit of this suggestion has been carried out for plates by Herrmann and Armenakas [2], who employed a variational method to derive their equations. An alternate procedure, which is used in the present paper, is to start with the already developed equations of non-linear elasticity and then to sequentially perturb the equations, integrate thru the thickness, and finally introduce displacement simplifications to obtain a linearized beam theory subjected to an initial non-uniform state of stress.† The advantages of this procedure are two-fold. Firstly it permits some physical understanding to be attached to the development of the governing equations and secondly it provides a check on the equations (one-dimensionalized) developed by the previous variational technique [2] and its associated assumptions. As will be seen the two sets of governing equations are at wide variance, and since the present technique is easily followed and checked one tends to believe the present results. The details of this procedure are developed in the following sections.

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† Obviously, this same technique may be used to generate a plate theory, which will be presented at a later date.

THE NON-LINEAR EQUATIONS OF ELASTICITY

Referring to Figure 1 and following accepted procedures [3], the stress vector equation of equilibrium of the deformed body is given by

$$\partial \vec{\sigma}_i^* / \partial x_i + \vec{X}^* = 0 \qquad (1)$$

where $\vec{\sigma}_i^*$ is the stress vector referred to the undeformed i^{th} face area and $\vec{X}^*$ is the body force vector referred to the undeformed volume. Resolving $\vec{\sigma}_i^*$ in the non-orthogonal lattice vector $(\vec{G}_j)$ directions yields

$$\vec{\sigma}_i^* = \sigma_{ij}^* \vec{G}_j \qquad (2)$$

where σ_{ij}^* are the Trefftz components of stress, referred to the undeformed i^{th} face area, which can be shown to be symmetric [3]. Since $\vec{G}_j = \frac{\partial \vec{R}}{\partial x_j}$ where $\vec{R}$ is the final state position vector it is seen that†

$$\vec{G}_j = (\delta_{js} + \frac{\partial u_s}{\partial x_j}) \vec{i}_s \qquad (3)$$

where $\vec{i}_s$ is the unit vector in the s^{th} orthogonal cartesian direction. Decomposing $\vec{X}^*$ into its $\vec{i}_s$ components

$$\vec{X}^* = X_s^* \vec{i}_s \qquad (4)$$

and putting the results of (2), (3), and (4) into (1) yields the following scalar equations of equilibrium.

$$\frac{\partial}{\partial x_i} \left[(\delta_{js} + \frac{\partial u_s}{\partial x_j}) \sigma_{ij}^* \right] + X_s^* = 0 \qquad (5)$$

---

† Note that $\vec{R} = \vec{r} + \vec{u} = (X_s + u_s) \vec{i}_s$, therefore $\frac{\partial \vec{R}}{\partial x_j} = (\delta_{js} + \frac{\partial u_s}{\partial x_j}) \vec{i}_s$

Referring to Figure 2 where $\vec{p}^*$ is the prescribed traction referred to the undeformed oblique face, and noting that $dA\ (\vec{i}_i \cdot \vec{n}) = dA_i/2$, the equilibrium of the deformed tetrahedron is given by

$$\vec{p}^* = (\vec{i}_i \cdot \vec{n})\ \vec{\sigma}_i^* \equiv n_i\ \vec{\sigma}_i^* \tag{6}$$

and by using (2) and (3) and defining the $\vec{i}_s$ components of $\vec{p}^*$ as p_s^*, (6) becomes

$$p_s^* = \sigma_{ij}^*\ n_i\ (\delta_{js} + \frac{\partial u_s}{\partial x_j}) \tag{7}$$

Equations (5) and (7) are the desired non-linear equations that describe the equilibrium condition and the surface tractions respectively. If the extensions and shears are small, the final areas and volume are equal respectively to the initial areas and volume so that $\sigma_{ij}^* = \sigma_{ij}$, $X_s^* = X_s$, and $p_s^* = p_s$ where σ_{ij}, X_s, and p_s are the actual stresses, body forces, and surface tractions respectively. These approximations are adopted for the work that follows, so that the equilibrium equations and the surface traction equations become,

$$\frac{\partial}{\partial x_i}\left[\ (\delta_{js} + \frac{\partial u_s}{\partial x_j})\ \sigma_{ij}\ \right] + X_s = 0 \tag{8}$$

$$p_s = \sigma_{ij}\ n_i\ (\delta_{js} + \frac{\partial u_s}{\partial x_j}) \tag{9}$$

THE PERTURBED EQUATIONS

Following a technique described by Bolotin [4] the following quantities are introduced

$$\tilde{u}_s = u_s + \bar{u}_s \tag{10}$$

$$\tilde{\sigma}_{ij} = \sigma_{ij} + \bar{\sigma}_{ij} \tag{11}$$

$$\tilde{p}_s = p_s + \Delta\, p_s + \bar{p}_s \tag{12}$$

$$\tilde{X}_s = X_s + \Delta\, X_s + \bar{X}_s - \rho \frac{\partial^2 \bar{u}_s}{\partial t^2} \tag{13}$$

where, for example, $\tilde{u}_s$, u_s, and $\bar{u}_s$ represent the final displacement, the initial displacement, and the perturbation displacement, respectively. The terms $\Delta\, p_s$ and $\Delta\, X_s$ represent the change in the initial tractions and body forces, respectively, due to the perturbation displacement. Since the final equilibrium equations and traction expressions have forms similar to (8) and (9) one has

$$\frac{\partial}{\partial x_i} \left[\left(\delta_{js} + \frac{\partial \tilde{u}_s}{\partial x_j}\right) \tilde{\sigma}_{ij} \right] + \tilde{X}_s = 0 \tag{14}$$

$$\tilde{p}_s = \tilde{\sigma}_{ij}\, n_i \left(\delta_{js} + \frac{\partial \tilde{u}_s}{\partial x_j}\right) \tag{15}$$

and upon entering (10) thru (13) into (14) and (15), and using the results (8) and (9) one has

$$\frac{\partial}{\partial x_i} \left(\sigma_{ij} \frac{\partial \bar{u}_s}{\partial x_j}\right) + \frac{\partial}{\partial x_i} \left[\bar{\sigma}_{ij} \left(\delta_{sj} + \frac{\partial u_s}{\partial x_j} + \frac{\partial \bar{u}_s}{\partial x_j}\right) \right] + \bar{X}_s + \Delta\, X_s - \rho \frac{\partial^2 \bar{u}_s}{\partial t^2} = 0 \tag{16}$$

$$\bar{p}_s + \Delta\, p_s = \left[\sigma_{ij} \frac{\partial \bar{u}_s}{\partial x_j} + \bar{\sigma}_{ij} \left(\delta_{js} + \frac{\partial u_s}{\partial x_j} + \frac{\partial \bar{u}_s}{\partial x_j}\right) \right] n_i \tag{17}$$

Linearizing the perturbation quantities, one neglects the term $\bar{\sigma}_{ij} \frac{\partial \bar{u}_s}{\partial x_j}$, and assuming the initial displacement gradients to be small [4] one neglects the term $\frac{\partial u_s}{\partial x_j}$. This results in the following set of relations.

$$\frac{\partial}{\partial x_i} \left(\sigma_{ij} \frac{\partial \bar{u}_s}{\partial x_j}\right) + \frac{\partial}{\partial x_i} (\bar{\sigma}_{is}) + \bar{X}_s + \Delta\, X_s - \rho \frac{\partial^2 \bar{u}_s}{\partial t^2} = 0 \tag{18}$$

$$\bar{p}_s + \Delta\, p_s = \left[\sigma_{ij} \frac{\partial \bar{u}_s}{\partial x_j} + \bar{\sigma}_{is} \right] n_i \tag{19}$$

THE BEAM PERTURBATION EQUATIONS

Assuming a two-dimensional problem (plane stress) in the $x_1 - x_3$ plane (18) and (19) become upon expansion[†]

$$\frac{\partial}{\partial x_1}\left(\sigma_{11}\frac{\partial \bar{u}_1}{\partial x_1}\right) + \frac{\partial}{\partial x_1}\left(\sigma_{13}\frac{\partial \bar{u}_1}{\partial x_3}\right) + \frac{\partial}{\partial x_3}\left(\sigma_{31}\frac{\partial \bar{u}_1}{\partial x_1}\right) + \frac{\partial}{\partial x_3}\left(\sigma_{33}\frac{\partial \bar{u}_1}{\partial x_3}\right) + \frac{\partial \bar{\sigma}_{11}}{\partial x_1} + \frac{\partial \bar{\sigma}_{31}}{\partial x_3}$$

$$+ \bar{X}_1 + \Delta X_1 - \rho \frac{\partial^2 \bar{u}_1}{\partial t^2} = 0 \qquad (20)$$

$$\frac{\partial}{\partial x_1}\left(\sigma_{11}\frac{\partial \bar{u}_3}{\partial x_1}\right) + \frac{\partial}{\partial x_1}\left(\sigma_{13}\frac{\partial \bar{u}_3}{\partial x_3}\right) + \frac{\partial}{\partial x_3}\left(\sigma_{31}\frac{\partial \bar{u}_3}{\partial x_3}\right) + \frac{\partial}{\partial x_3}\left(\sigma_{33}\frac{\partial \bar{u}_3}{\partial x_3}\right) + \frac{\partial \bar{\sigma}_{13}}{\partial x_1} + \frac{\partial \bar{\sigma}_{33}}{\partial x_3}$$

$$+ \bar{X}_3 + \Delta X_3 - \rho \frac{\partial^2 \bar{u}_3}{\partial t^2} = 0 \qquad (21)$$

$$\bar{p}_1 + \Delta p_1 = \sigma_{11}\frac{\partial \bar{u}_1}{\partial x_1} + \sigma_{13}\frac{\partial \bar{u}_1}{\partial x_3} + \bar{\sigma}_{11} \qquad (22)$$

$$\bar{p}_3 + \Delta p_3 = \sigma_{11}\frac{\partial \bar{u}_3}{\partial x_1} + \sigma_{13}\frac{\partial \bar{u}_3}{\partial x_3} + \bar{\sigma}_{13} \qquad (23)$$

Now multiply (20) - (23) by dx_3 and integrate thru the initial thickness[††]; then multiply (20) and (22) by $x_3 dx_3$ and again integrate thru the thickness. Into these results insert the following displacement and stress assumptions,

$$\bar{u}_1 (x_1, x_3, t) = u (x_1, t) + x_3 \psi (x_1, t) \qquad (24)$$

$$\bar{u}_3 (x_1, x_3, t) = w (x_1, t) \qquad (25)$$

$$\bar{\sigma}_{11} = E \frac{\partial \bar{u}_1}{\partial x_1} = E (u' + x_3 \psi') \qquad (26)$$

$$\bar{\sigma}_{13} = \kappa^2 G \left(\frac{\partial \bar{u}_1}{\partial x_3} + \frac{\partial \bar{u}_3}{\partial x_1}\right) = \kappa^2 G (\psi + w') \qquad (27)$$

to obtain the following beam equations of motion and associated boundary conditions.

---

† Note that boundary conditions are desired only on the x_1 faces, therefore $n_1 = 1$, $n_2 = n_3 = 0$.

†† that is, the limits of all integrals are from - h/2 to + h/2.

$$(u' N_x)' + (\psi' M_x)' + (\psi Q_x)' + E h u'' + f_x = \rho h \ddot{u} \tag{28}$$

$$(w' N_x)' + \kappa^2 G h (\psi' + w'') + q = \rho h \ddot{w} \tag{29}$$

$$(u' M_x + \psi' M_*)' + (\psi Q_*)' + \frac{Eh^3}{12} \psi'' - u' Q_x - \psi' Q_* - \psi N_z - \kappa^2 G h (\psi + w') + m_x = \rho \frac{h^3}{12} \ddot{\psi} \tag{30}$$

$$\overline{F}_1 + \Delta F_1 = u' N_x + \psi' M_x + \psi Q_x + E h u' \qquad \text{or } u = \overline{u}_0{}^{\dagger} \tag{31}$$

$$\overline{F}_3 + \Delta F_3 = w' N_x + \kappa^2 G h (\psi + w') \qquad \text{or } w = \overline{w}_0 \tag{32}$$

$$\overline{M}_1 + \Delta M_1 = u' M_x + \psi' M_* + \psi Q_* + E \frac{h^3}{12} \psi' \qquad \text{or } \psi = \overline{\psi}_0 \tag{33}$$

where $\overline{u}_0$, $\overline{w}_0$, and $\overline{\psi}_0$ have been used to denote prescribed displacements, primes denote differentiation with respect to x_1, dots denote differentiation with respect to time, and where the following definitions have been used.

$$f_x = \int (\overline{X}_1 + \Delta X_1) dx_3 \quad + \sigma_{31}{}^{(+)} (u' + \frac{h}{2} \psi') - \sigma_{31}{}^{(-)} (u' - \frac{h}{2} \psi')$$
$$+ [\, \sigma_{33}{}^{(+)} - \sigma_{33}{}^{(-)}\,]\, \psi + \overline{\sigma}_{31}{}^{(+)} - \overline{\sigma}_{31}{}^{(-)}$$

$$q = \int (\overline{X}_3 + \Delta X_3) dx_3 \quad + [\, \sigma_{31}{}^{(+)} - \sigma_{31}{}^{(-)}\,]\, w' + \overline{\sigma}_{33}{}^{(+)} - \overline{\sigma}_{33}{}^{(-)}$$

$$m_x = \int (\overline{X}_1 + \Delta X_1) x_3 \, dx_3 \quad + \frac{h}{2} [\, \sigma_{31}{}^{+} (u' + \frac{h}{2} \psi') + \sigma_{31}{}^{(-)} (u' - \frac{h}{2} \psi')$$
$$+ (\sigma_{33}{}^{(+)} + \sigma_{33}{}^{(-)})\, \psi + \overline{\sigma}_{31}{}^{+} + \overline{\sigma}_{31}{}^{(-)}\,]$$

$$\overline{F}_i = \int \overline{p}_i \, dx_3 \qquad (i = 1,3) \qquad N_x = \int \sigma_{11} \, dx_3$$

$$\Delta F_i = \int \Delta p_i \, dx_3 \qquad (i = 1,3) \qquad N_z = \int \sigma_{33} \, dx_3$$

† Note that there is a choice of stress or displacement boundary conditions.

$\bar{M}_1 = \int \bar{p}_1 \, x_3 \, dx_3$ $\qquad$ $Q_x = \int \sigma_{13} \, dx_3$

$\Delta M_1 = \int \Delta p_1 \, x_3 \, dx_3$ $\qquad$ $Q_* = \int \sigma_{31} \, x_3 \, dx_3$

$M_x = \int \sigma_{11} \, x_3 \, dx_3$ $\qquad$ $M_* = \int \sigma_{11} \, x_3^2 \, dx_3$

Note that the notation $\sigma_{33}{}^{(+)} \equiv \sigma_{33}\,(h/2)$ and $\bar{\sigma}_{33}{}^{(-)} \equiv \bar{\sigma}_{33}\,(-\,h/2)$ etc. In the following sections the above results are used to analyze some stability and vibration problems.

STABILITY PROBLEMS

This section employs static methods (i.e. the "Euler Method") to determine the stability of the various beam configurations considered. This is an adequate approach if the loadings are conservative. Upon careful examination it can be shown that the loading in Case III is non-conservative, hence the confirmation of that result must be postponed until the corresponding dynamic formulation of that problem is carried out in the next section entitled "VIBRATION PROBLEMS".

Case I, Just $\sigma_{11} = \sigma_N$† Acting

Equations (28) thru (33) reduce†† to

$$(\sigma_N + E)\, u'' = 0 \tag{34}$$

$$(\sigma_N + \kappa^2 G)\, w'' + \kappa^2 G\, \psi' = 0 \tag{35}$$

$$(\sigma_N + \frac{Eh^2}{12})\, \psi'' - \kappa^2 G\, (\psi + w') = 0 \tag{36}$$

$$u\,(y) = \psi'\,(y) = w\,(y) = 0 \quad ; \quad y = o, \ell \tag{37}$$

---

† σ_N is a constant

†† In all following examples, the boundary conditions are chosen to yield the simplest solutions.

Neglecting (34), which leads to trivial results, it is assumed that

$$w = W \sin \pi x/\ell$$

$$\psi = \Psi \cos \pi x/\ell$$

whence a non-trivial solution for W and Ψ demands that $\det |a_{ij}| = 0$ where

$$a_{11} = (S\sigma - 1)\pi/\ell \qquad a_{12} = -1$$

$$a_{21} = -\pi/\ell \qquad a_{22} = (\sigma - K)(S/K) - 1$$

and

$$S = (E/G)(h/\ell)^2 \quad K = (\ell/\kappa h)^2 \quad \kappa^2 = \pi^2/12 \quad \sigma = -\sigma_N K/E \equiv \frac{-N_x}{(N_x)_{Euler}}$$

Hence the following equation for σ is obtained where the lowest root is desired

$$S\sigma^2 - (SK + K + 1)\sigma + K = 0 \qquad (38)$$

When S=0 (i.e. $G \to \infty$) the result reduces to $\sigma_0 = K/(K + 1)$. Since usually $K > 100$ this result is seen to be close to the result obtained from a Bernoulli-Euler beam analysis ($\sigma_0 = 1$). For $S \geq 0$, the non-dimensional buckling stress σ is shown in Figure 3 where it is compared with the classical result † obtained in Reference 5. The close agreement justifies the use of the simpler analysis developed in Reference 5.

Case II, Just†† $\sigma_{11} = 2 x_3 \sigma_M/h$

Equations (28) thru (33) reduce to

$$E h u'' + M_x \psi'' = 0 \qquad (39)$$

$$\kappa^2 G h (\psi' + w'') = 0 \qquad (40)$$

† The classical result is $\sigma = (1 + S)^{-1}$
†† σ_M is a constant

$$\frac{Eh^3}{12}\psi'' + M_x u'' - \kappa^2 G h (\psi + w') = 0 \tag{41}$$

$$M_x \psi'(y) + E h u'(y) = M_x u'(y) + \frac{Eh^3}{12}\psi'(y) = w(y) = 0 \quad ; y = 0, \ell \tag{42}$$

where $M_x = 2 h^2 \sigma_M/12$. Assuming that,

$$w = W \sin \pi x/\ell$$

$$\psi = \Psi \cos \pi x/\ell$$

$$u = U \cos \pi x/\ell$$

a non-trivial solution for W, Ψ, and U demands that det $|a_{ij}| = 0$ where

$a_{11} = E h$ $\quad a_{12} = 0$ $\quad a_{13} = M_x$

$a_{21} = 0$ $\quad a_{22} = \kappa^2 G h (\pi/\ell)$ $\quad a_{23} = \kappa^2 G h$

$a_{31} = M_x (\pi/\ell)^2$ $\quad a_{32} = \kappa^2 G h (\pi/\ell)$ $\quad a_{33} = \kappa^2 G h + E h^3 (\pi/\ell)^2/12$

Expanding det $|a_{ij}| = 0$ yields the following result,

$$\sigma_M = \pm \sqrt{3}\ E \tag{43}$$

Thus a buckling stress is predicted, even though it occurs at a stress which is larger than that permissible for linear elasticity. However, a simple plastic yield stress may be obtained by substituting the tangent modulus E_t for E in (43).

Case III, Just $\sigma_{13} = \sigma_Q$† Acting

Equations (28) thru (33) reduce to

$$2 \sigma_Q \psi' + E u'' = 0 \tag{44}$$

$$\kappa^2 G h (\psi' + w'') = 0 \tag{45}$$

† σ_Q is a constant

$$\frac{Eh^2}{12} \psi'' - \kappa^2 G (\psi + w') = 0 \tag{46}$$

$$u (y) = \psi' (y) = w (y) = 0 \quad ; y = 0, \ell \tag{47}$$

Assuming that,

$$u = U \sin \pi x / \ell$$

$$\psi = \Psi \cos \pi x / \ell$$

$$w = W \sin \pi x / \ell$$

a non-trivial solution for W, Ψ, and U demands that det $|a_{ij}| = 0$ where

$$a_{11} = E(\pi/\ell) \qquad a_{12} = 0 \qquad a_{13} = 2 \sigma_Q$$

$$a_{21} = 0 \qquad a_{22} = \pi/\ell \qquad a_{23} = 1$$

$$a_{31} = 0 \qquad a_{32} = \kappa^2 G (\pi/\ell) \qquad a_{33} = \kappa^2 G + E h^2 (\pi/\ell)^2/12$$

Expanding the det $|a_{ij}|$ yields

$$\det |a_{ij}| = E^2 h^2 \kappa^2 (\pi/\ell)^6 G/12 \neq 0 \tag{48}$$

hence (48) demonstrates that a buckling stress <u>does not exist</u>.† The next case demonstrates, however, that if σ_Q is applied conservatively, a buckling stress <u>does exist</u>.

<u>Case IV, Just $\sigma_{13} = \sigma_Q$ Acting Conservatively</u>

In this case $f_x = m_x \equiv 0$ so that (28) thru (33) reduce to

$$\sigma_Q \psi' + E u'' = 0 \tag{49}$$

† The confirmation of this result is found in Case IV of the next section entitled "VIBRATION PROBLEMS".

$$\kappa^2 G h (\psi' + w'') = 0 \tag{50}$$

$$\frac{h^2}{12} E \psi'' - \sigma_Q u' - \kappa^2 G (\psi + w') = 0 \tag{51}$$

$$u (y) = \psi' (y) = w (y) = 0 \quad ; \quad y = 0,\ell \tag{52}$$

Assuming that

$$u = U \sin \pi x/ \ell$$

$$\psi = \Psi \cos \pi x/ \ell$$

$$w = W \sin \pi x/ \ell$$

a non-trivial solution for W, Ψ, and U demands that det $|a_{ij}| = 0$ where

$$a_{11} = K (\pi/\ell) \qquad a_{12} = 0 \qquad a_{13} = K \sigma_Q/E$$

$$a_{21} = 0 \qquad a_{22} = (\pi/\ell)/S \qquad a_{23} = 1/S$$

$$a_{31} = K \sigma_Q (\pi/\ell)/E \qquad a_{32} = (\pi/\ell)/S \qquad a_{33} = 1 + 1/S$$

Expanding det $|a_{ij}| = 0$ yields the following result,

$$\sigma_Q = \pm E \kappa (h/\ell) \tag{53}$$

This result has been determined previously by Herrmann and Armenakas [2]. However, as demonstrated by Case III, their statement "... we could assume that the resulting shear forces either rotate with the element or do not change direction after deformation." can not be correct.

VIBRATION PROBLEMS

Case I, Longitudinal Vibration with Just $\sigma_{11} = \sigma_N$ Acting

In this case (28) and (31) are uncoupled and become

$$(\sigma_N + E)\ u'' = \rho\ \ddot{u} \tag{54}$$

$$u\ (y,t) = 0 \qquad ; \quad y = 0,\ell \tag{55}$$

Assuming $u = U_n\ (\sin n\ \pi\ x/\ \ell)\ e^{i\omega_n t}$ and letting $\sigma_N = \pm\ \alpha\ E$ the solution is given by

$$\frac{\omega_n}{(\omega_n)_{cl}} = (1 \pm \alpha)^{1/2} \tag{56}$$

where

$$(\omega_n)_{cl} = (E/\rho)^{1/2}\ n\ \pi/\ \ell$$

and

$$n = 1,\ 2,\ 3,\ \ldots$$

For some materials (glass fibers, for example) α may be as large as .1, hence for this value of α it is seen that (56) may be in the range from .95 to 1.05 for all values of n.

Case II, Transverse Vibration with Just $\sigma_{11} = \sigma_N$ Acting

In this case just (29), (30) and (32), (33) are coupled and become

$$(\sigma_N + \kappa^2\ G)\ w'' + \kappa^2\ G\ \psi' = \rho\ \ddot{w} \tag{57}$$

$$(\sigma_N + E)\ \frac{h^2}{12}\ \psi'' - \kappa^2\ G\ (\psi + w') = \rho\ \frac{h^2}{12}\ \ddot{\psi} \tag{58}$$

$$\psi'\ (y,t) = w\ (y,t) = 0 \qquad ; \quad y = 0,\ell \tag{59}$$

Assuming that,

$$w = W_n\ (\sin n\ \pi\ x/\ \ell)\ e^{i\omega_n t}$$

$$\psi = \Psi_n\ (\cos n\ \pi\ x/\ \ell)\ e^{i\omega_n t}$$

a non-trivial solution for W_n and Ψ_n demands that det $|a_{ij}| = 0$ where

$$a_{11} = n^2 (S\sigma - 1) + \Omega^2_n S \qquad a_{12} = -n\ell/\pi$$

$$a_{21} = -n K \pi/\ell \qquad a_{22} = n^2 S (\sigma - K) - K + S\Omega^2_n$$

and

$$\Omega^2_n = \rho K (\ell/\pi)^2 \omega^2_n/E$$

The frequency spectrum thus is given by

$$\Omega_n = [(-B \pm [B^2 - 4AC]^{1/2})/2A]^{1/2} \tag{60}$$

where

$$A = S$$

$$B = n^2 \sigma S + n^2 S (\sigma - K) - K - n^2$$

$$C = n^2 \sigma [n^2 S (\sigma - K) - K] - n^4 (\sigma - K)$$

When $S \to 0$ (zero transverse shear) (60) specializes to

$$\Omega_n = [(-n^2 \sigma K + n^4 [K - \sigma])/[K + n^2]]^{1/2} \tag{61}$$

Note that when $\sigma = S = 0$ and K is large, $\Omega_n \cong n^2$ which is the classical result. Figure 4 plots the first two non-dimensional frequencies versus σ for three values of E/G† with $\ell/h = 10$. The classical values [5] are shown by circles and are in very good agreement with the present results, thus completely confirming the results obtained in Reference 5.

Case III, Vibration with Just $\sigma_{11} = 2 x_3 \sigma_M/h$ Acting

Equations (28) thru (33) reduce to

† Note that E/G = 0 yields the classical values and that E/G = 2.6 yields the isotropic values.

$$M_x \ddot{\psi} + E\,h\,u'' = \rho\,h\,\ddot{u} \tag{62}$$

$$\kappa^2 G\,h\,(\psi' + w'') = \rho\,h\,\ddot{w} \tag{63}$$

$$M_x\,u'' + \frac{Eh^3}{12}\psi'' - \kappa^2 G\,h\,(\psi + w') = \rho\,\frac{h^3}{12}\ddot{\psi} \tag{64}$$

$$M_x\,(y,t) + E\,h\,u'\,(y,t) = M_x\,u'\,(y,t) + \frac{Eh^3}{12}\psi'\,(y,t)$$

$$= w\,(y,t) = 0 \quad ; \quad y = 0,\ell \tag{65}$$

Assuming that

$$w = W_n\,(\sin n\,\pi\,x/\,\ell)\,e^{i\omega_n t}$$
$$\psi = \Psi_n\,(\cos n\,\pi\,x/\,\ell)\,e^{i\omega_n t}$$
$$u = U_n\,(\cos n\,\pi\,x/\,\ell)\,e^{i\omega_n t}$$

a non-trivial solution for W_n, Ψ_n, and U_n demands that det $|a_{ij}| = 0$ where $\hat{\sigma} = K\,\sigma_M/E$ and

$$a_{11} = K\,n^2 - \Omega^2_n \qquad a_{12} - 0 \qquad a_{13} = 2\,n^2\,\ell^2\,\hat{\sigma}/K\,h\,\pi^2$$
$$a_{21} = 0 \qquad a_{22} = n^2 - \Omega^2_n\,S \qquad a_{23} = n\,\ell/\pi$$
$$a_{31} = 2\,n^2\,S\,\hat{\sigma}/h \qquad a_{32} = n\,\pi\,K/\ell \qquad a_{33} = K\,(1 + n^2\,S) - S\,\Omega^2_n$$

Expansion of the determinant yields the following expression for Ω_n.

$$(K\,n^2 - \Omega^2_n)\,[\,n^4\,K - \Omega^2_n\,(n^2 + K + n^2\,K\,S) - S\,\Omega^4_n\,] + (\Omega^2_n\,S - n^2)\,(n^2\,\hat{\sigma})^2/3 = 0 \tag{66}$$

For zero transverse shear $S \to 0$ and (66) reduces to

$$(n^2 + K)\,\Omega^4_n - K\,n^2\,(2\,n^2 + K)\,\Omega^2_n + n^6\,(K^2 - \hat{\sigma}^2/3) = 0 \tag{67}$$

Noting that $\hat{\sigma}$ appears only in the zeroth power term in (66) and (67) and noting that $\sigma_M < .1\,E$ (hence $\hat{\sigma} < .1\,K$) it is seen that $\hat{\sigma}$ has a negligible effect on the frequencies.

Case IV, Vibration with Just $\sigma_{13} = \sigma_Q$ Acting

Equations (28) thru (33) reduce to

$$2\,\sigma_Q\,\psi' + E\,u'' = \rho\,\ddot{u} \tag{68}$$

$$\kappa^2\,G\,(\psi' + w'') = \rho\,\ddot{w} \tag{69}$$

$$E\,\frac{h^2}{12}\,\psi'' - \kappa^2\,G\,(\psi + w') = \rho\,\frac{h^2}{12}\,\ddot{\psi} \tag{70}$$

$$u\,(y,t) = \psi'\,(y,t) = w\,(y,t) = 0 \quad ; \; y = 0,\ell \tag{71}$$

Assuming that

$$u = U_n\,(\sin n\,\pi\,x/\,\ell)\,e^{i\omega_n t}$$

$$\psi = \Psi_n\,(\cos n\,\pi\,x/\,\ell)\,e^{i\omega_n t}$$

$$w = W_n\,(\sin n\,\pi\,x/\,\ell)\,e^{i\omega_n t}$$

a non-trivial solution for U_n, Ψ_n, and W_n demands that det $|a_{ij}| = 0$ where $\sigma_* = -\,K\,\sigma_Q/E$ and

$$a_{11} = \Omega^2_n - K\,n^2 \qquad a_{12} = 0 \qquad a_{13} = -\,2\,\sigma_*\,n\,\ell/\pi$$

$$a_{21} = 0 \qquad a_{22} = S\,\Omega^2_n - n^2 \qquad a_{23} = -\,n\,\ell/\pi$$

$$a_{31} = 0 \qquad a_{32} = -\,n\,\pi\,K/\ell \qquad a_{33} = S\,\Omega^2_n - K\,(1 + n^2\,S)$$

Expansion of the determinant, given below, demonstrates that σ_* does not affect any of the frequencies† and in fact the longitudinal frequency is uncoupled from the other two coupled frequencies.

$$(\Omega^2_n - K\,n^2)\,[\,(S\,\Omega^2_n - n^2)\,(S\,\Omega^2_n - K\,[1 + n^2\,S]) - n^2\,K\,] = 0 \tag{72}$$

For zero transverse shear $S \to 0$ and (72) reduces to

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† In particular no value of σ_* can make the frequencies zero or complex, hence instability cannot occur as was predicted in Case III of the previous section entitled "STABILITY PROBLEMS".

$$(\Omega^2_n - K\,n^2)\,[\,S\,\Omega^4_n - (n^2 + K\,[1 + n^2\,S])\,\Omega^2_n + n^4\,K\,] = 0 \quad (73)$$

Case V, Vibration with Just $\sigma_{13} = \sigma_Q$ Acting Conservatively

In this case $f_x = m_x \equiv 0$ so that (28) thru (33) reduce to

$$\sigma_Q\,\psi' + E\,u'' = \rho\,\ddot{u} \qquad (74)$$

$$\kappa^2\,G\,(\psi' + w'') = \rho\,\ddot{w} \qquad (75)$$

$$E\,\frac{h^2}{12}\,\psi'' - \sigma_Q\,u' - \kappa^2\,G\,(\psi + w') = \rho\,\frac{h^2}{12}\,\ddot{\psi} \qquad (76)$$

$$u\,(y,t) = \psi'\,(y,t) = w\,(y,t) = 0 \quad ; \quad y = 0,\ell \qquad (77)$$

Assuming that

$$u = U_n\,(\sin n\,\pi\,x/\,\ell)\,e^{i\omega_n t}$$
$$\psi = \Psi_n\,(\cos n\,\pi\,x/\,\ell)\,e^{i\omega_n t}$$
$$w = W_n\,(\sin n\,\pi\,x/\,\ell)\,e^{i\omega_n t}$$

a non-trivial solution for U_n, Ψ_n, and W_n demands that det $|a_{ij}| = 0$ where

$$a_{11} = \Omega^2_n - K\,n^2 \qquad a_{12} = 0 \qquad a_{13} = \sigma_*\,n\,\ell/\pi$$
$$a_{21} = 0 \qquad a_{22} = S\,\Omega^2_n - n^2 \qquad a_{23} = -\,n\,\ell/\pi$$
$$a_{31} = n\,\pi\,S\,K\,\sigma_*/\ell \qquad a_{32} = -\,n\,K\,\pi/\ell \qquad a_{33} = S\,\Omega^2_n - K\,(1 + n^2\,S)$$

Expansion of the determinant yields the following expression for Ω_n.

$$(\Omega^2_n - K\,n^2)\,[\,S\,\Omega^4_n - K\,(1 + n^2\,S)\,\Omega^2_n - n^2\,\Omega^2_n + n^4\,K\,] - \sigma_*^2\,n^2\,K\,(S\,\Omega^2_n - n^2) = 0 \qquad (78)$$

For zero transverse shear $S \to 0$ and (78) reduces to

$$\Omega_n = \left[\left[- B \pm (B^2 - 4\ A\ C)^{1/2} \right] / 2\ A\ \right]^{1/2} \tag{79}$$

where

$$A = K + n^2$$

$$B = - \left[n^4\ K + n^2\ K\ (K + n^2) \right]$$

$$C = n^4\ K \left[n^2\ K - \sigma_*^2 \right]$$

Figure 5 plots the first two non-dimensional frequencies versus σ_* for three values of E/G with $\ell/h = 10$.

COMPARISON WITH A PREVIOUS THEORY

Referring to Reference 2 and one-dimensionalizing the results† yields the following equations to be compared with (28), (29), and (30).

$$(\psi\ Q_x)' + E\ h\ u'' + \overline{f}_x = \rho\ h\ \ddot{u} \tag{80}$$

$$(w'\ N_x)' + \kappa^2\ G\ h\ (\psi' + w'') + \overline{q} = \rho\ h\ \ddot{w} \tag{81}$$

$$\frac{Eh^3}{12}\ \psi'' - u'\ Q_x - \kappa^2\ G\ h\ (\psi + w') + \overline{m}_x = \rho\ \frac{h^3}{12}\ \ddot{\psi} \tag{82}$$

It is noticed that the form of (81) and (29) agree, but that the forms of (80) and (28) and of (82) and (30) have marked differences. Additionally, Reference 2 leaves the terms $\overline{f}_x$, $\overline{q}$, and $\overline{m}_x$ to be arbitrary whereas the present paper demands specific forms for f_x, q, and m_x as seen by the first three results following (33). Also in a previous criticism Masur [6] has pointed out that a term $(u'\ N_x)'$ should appear in (80) which is accounted for in the corresponding equation (28) of the present work. Finally, since

† Notational changes have been made so that the comparison of equations (36a), (36c), and (36d) of Reference 2 with (28), (29), and (30) of the present work may be facilitated.

the present work is based on simple straight-forward origins and is so easy to check it is difficult to imagine that it can be in error; hence one is forced to the conclusion that the one-dimensionalized results of Reference 2 are not correct.

CONCLUSIONS

The several conclusions that may be inferred from this investigation are that (a) with just σ_N acting, the simpler work of Reference 5 is adequate to describe the problem, (b) with just σ_Q acting, very small changes in the direction of loading exert a large influence on determining whether or not the system can be unstable, (c) with just σ_M acting negligible effects are found, (d) transverse isotropy still exerts a significant influence on stability and vibration regardless of the type of initial stresses that are acting, and finally (e) the one-dimensionalized version of a previous work [2] by Herrmann and Armenakas does not accurately describe the present problem, hence doubt is cast as to the validity of their paper in general.

NOMENCLATURE

E	longitudinal Young's modulus
$\overline{F}_i + \Delta F_i$; $i = 1,3$	perturbation boundary forces
f_x, q, m_x	perturbation loadings
G	transverse shear modulus
h	beam thickness
$K = (\ell/\kappa h)^2$	geometry parameter
ℓ	beam length
$\overline{M}_1 + \Delta M_1$	perturbation boundary moment
N_x, N_z, Q_x, Q_*, M_x, M_*	initial generalized stress resultants
$\vec{p}^*$	generalized traction vector
p_s	actual traction components
$\overline{p}_i + \Delta p_i$	perturbation boundary stresses
$S = (E/G)\ (h/\ell)^2$	transverse isotropy parameter
u, w	perturbation displacements
$\overline{u}_0$, $\overline{w}_0$	perturbation displacements at the boundary
$\vec{X}^*$	generalized body force vector
X_s	actual body force components
$(\tilde{\ })$	final quantities
$(\overline{\ })$	perturbation quantities
$\kappa^2 = \pi^2/12$	Mindlin's shear correction factor
ψ	perturbation rotation
$\overline{\psi}_0$	perturbation rotation at the boundary
$\vec{\sigma}_i^*$	generalized stress vector
$\overline{\sigma}_{11}$, $\overline{\sigma}_{13}$	perturbation stresses
$\sigma_* = -K\,\sigma_Q/E$	non-dimensionalized shear stress
$\hat{\sigma} = K\,\sigma_M/E$	non-dimensionalized outer fiber stress

$\sigma = -\sigma_N K/E$	non-dimensionalized normal stress
$\sigma_N, \sigma_Q, \sigma_M$	constant initial stresses
σ_{ij}^*	Trefftz components of stress
σ_{ij}	actual stresses
Ω_n	non-dimensional frequency

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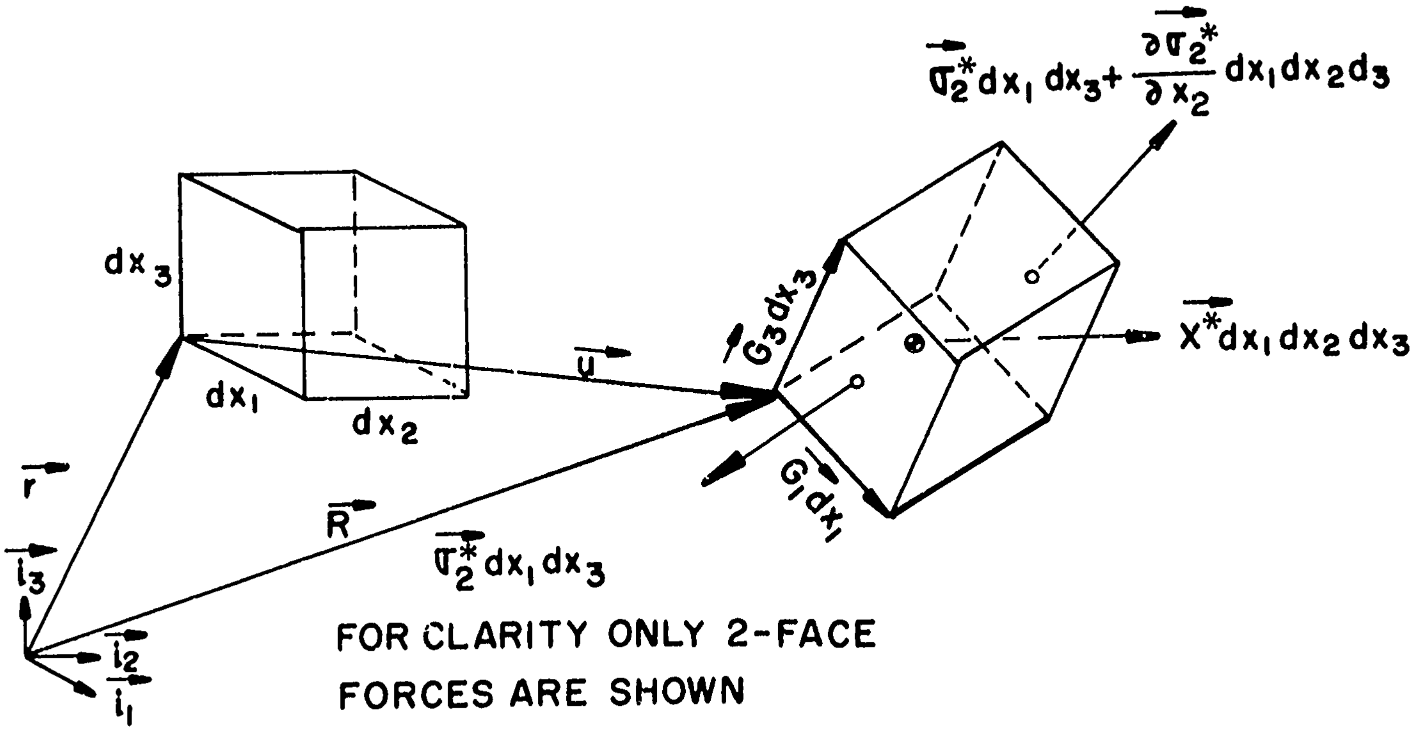

FIGURE I

EQUILIBRIUM OF A DEFORMED ELEMENT

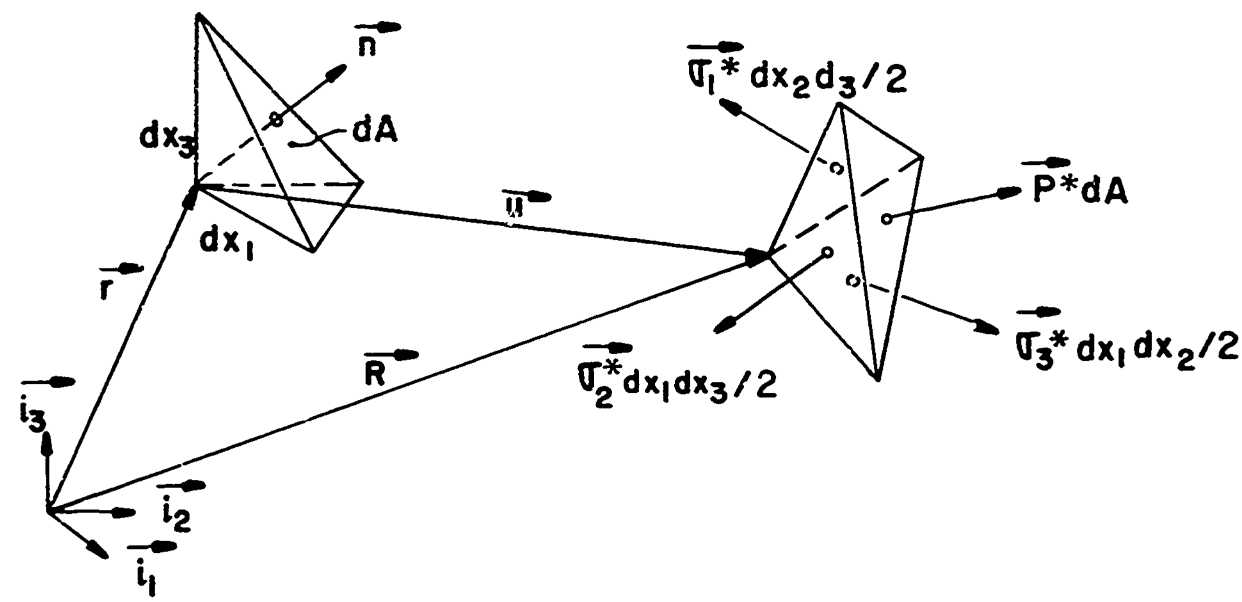

FIGURE 2

EQUILIBRIUM WITH THE SURFACE TRACTION

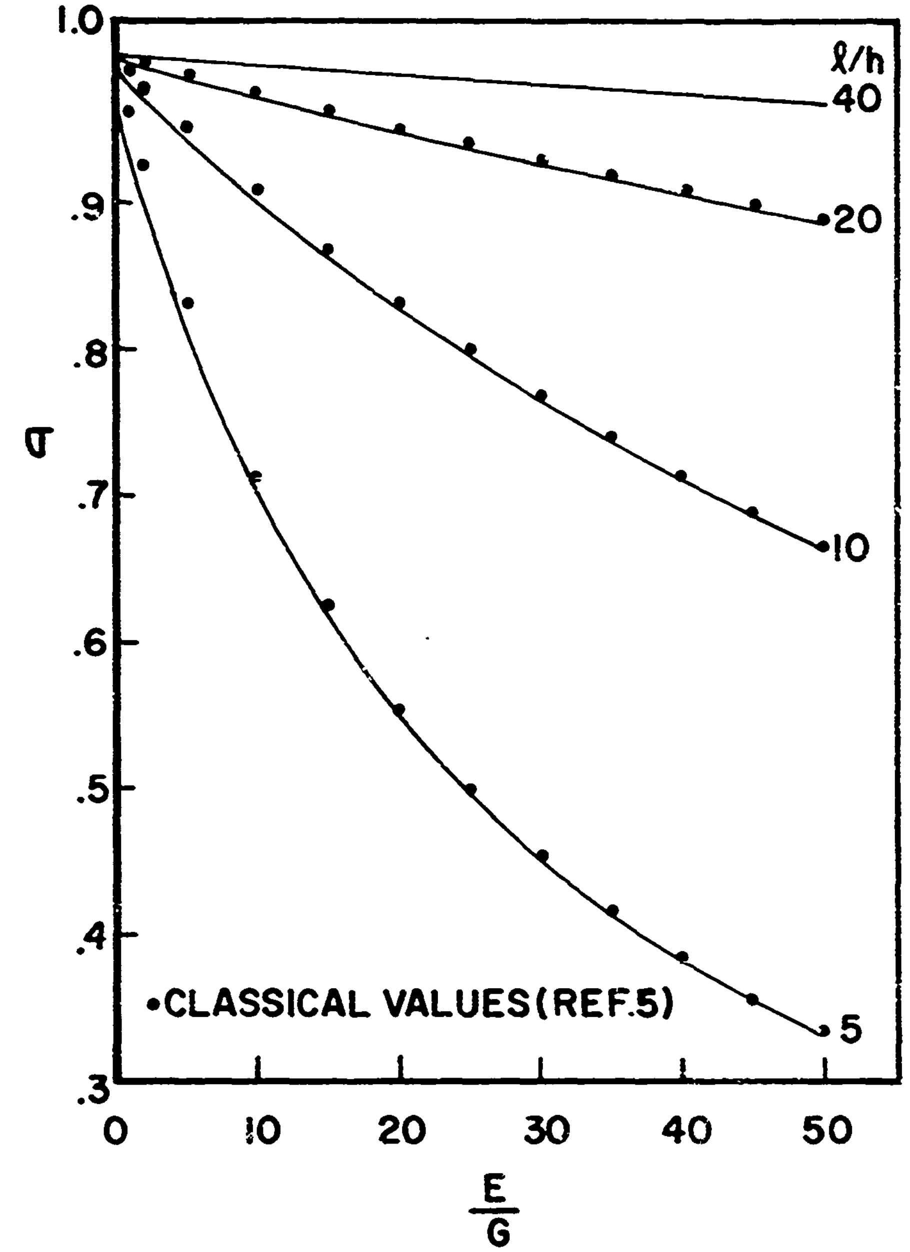

FIGURE 3

NON-DIMENSIONAL BUCKLING STRESS σ VERSUS E/G

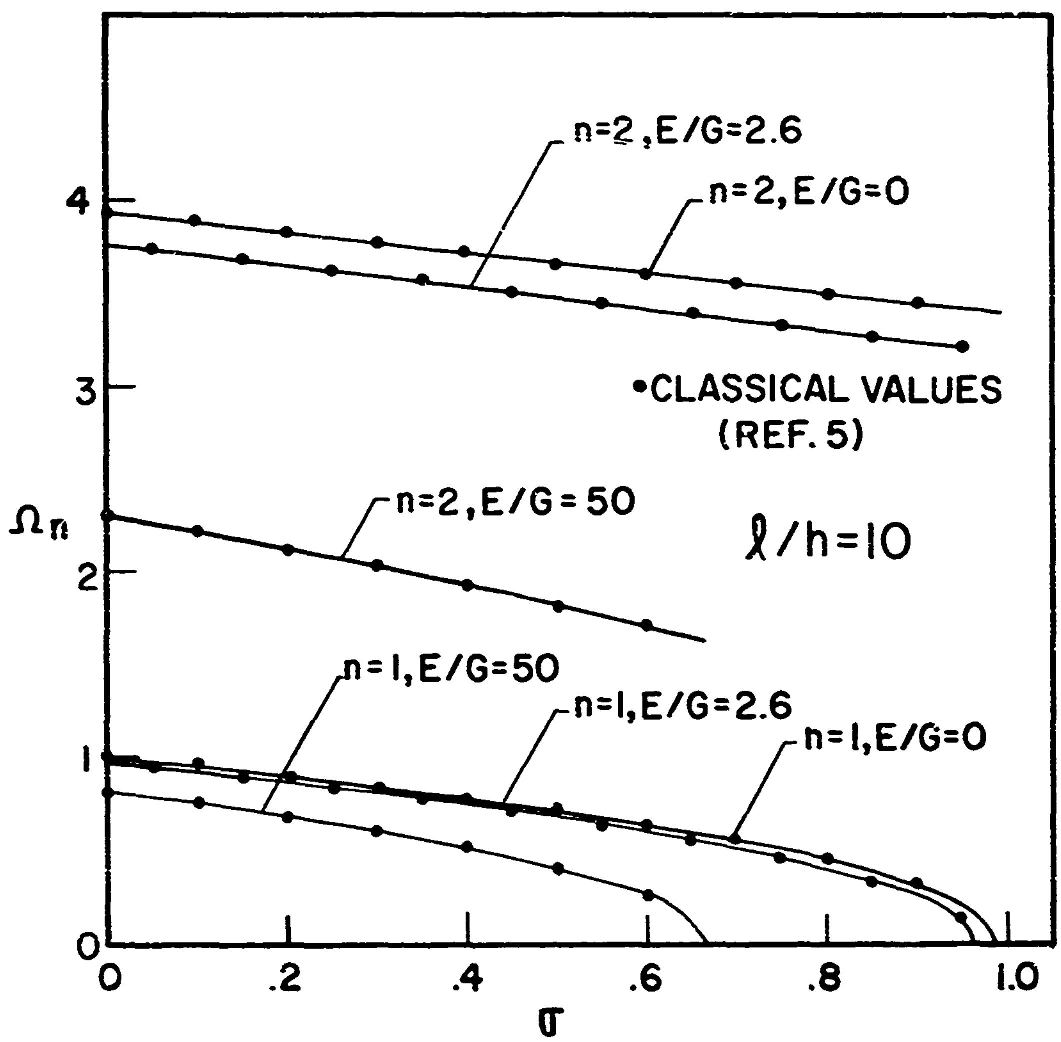

FIGURE 4

NON-DIMENSIONAL FREQUENCY Ω_n VERSUS σ

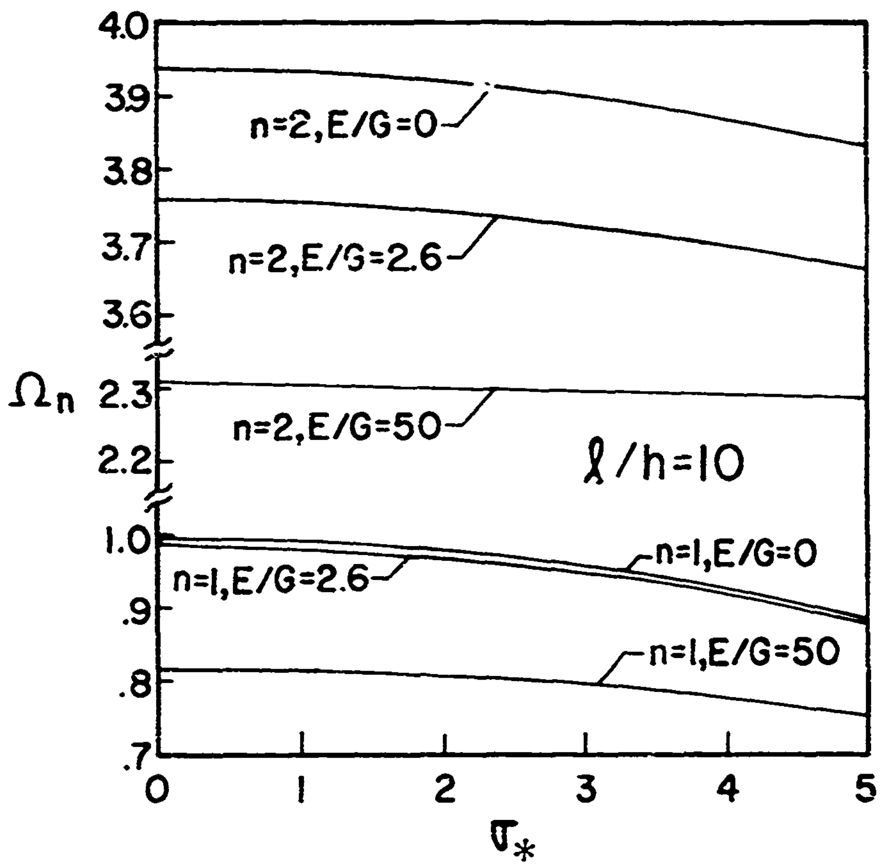

FIGURE 5

NON-DIMENSIONAL FREQUENCY Ω_n VERSUS σ_*